Magnetized Black Hole on Taub-Nut Instanton

Petya G. Nedkova[1][*], Stoytcho S. Yazadjiev[1,2][†]

[1] Department of Theoretical Physics, Faculty of Physics, Sofia University,

5 James Bourchier Boulevard, Sofia 1164, Bulgaria

[2] Theoretical Astrophysics, Eberhard Karls University of Tübingen,

Auf der Morgenstelle 10, 72076 Tübingen, Germany

Abstract

We present an exact solution to the 5D Einstein-Maxwell-dilaton equations describing a static black hole on Taub-Nut instanton. By construction the solution does not possess a charge, but is magnetized along the compact dimension. As a limit we obtain a new regular solution representing a magnetized Kaluza-Klein monopole. We investigate the relevant physical properties and derive the Smarr-like relations.

1 Introduction

In recent years it has been demonstrated that higher dimensional gravity admits a variety of solutions with nontrivial geometry. Among the different configurations, certain black holes were constructed, which were described as 'sitting on' gravitational instantons. This class of solutions includes the so called black holes on Kaluza-Klein bubbles [1]-[7], as well as black holes on Taub-Nut, Taub-Bolt, Kerr- and Eguchi-Hanson instantons [8]-[13]. A recent review on the topic is available written by Chen and Teo [12].

The instanton solutions possess interesting physical properties induced by their complicated geometry. In a recent work we discussed the thermodynamics of vacuum and electrostatic black holes on asymptotically locally flat gravitational instantons [14]. The goal of the current paper is to achieve some progress in the investigation of their behavior in magnetic fields.

Magnetized black holes have attracted a lot of attention in astrophysics since it is considered that they can provide viable models for realistic stellar-mass and supermassive black holes. Different mechanisms of electromagnetic energy extraction from rotating magnetized black hole have been proposed, the Blandford-Znajek one considered the most relevant [15], hoping to explain the formation of the highly relativistic

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[*]E-mail:pnedkova@phys.uni-sofia.bg

[†]E-mail: yazad@phys.uni-sofia.bg

jets from galactic nuclei. Other interesting physical phenomena were discovered as well concerning black holes in magnetic fields [16]. Such is the gravitational analog of the Meissner effect which consists in the expulsion of the magnetic flux lines from black holes horizons as they approach extremality [17]-[19], and the charge accretion leading to the charging up of a rotating black holes immersed in external magnetic field [20]-[22]. The scattering and Hawking radiation of magnetized black holes were also actively investigated, as well as the motion of charged particles in their vicinity [23]-[25]. It was demonstrated that the super-radiant instability exhibited by rotating black holes and the intensity of the Hawking evaporation is amplified in the presence of magnetic field [26], [27]. Very recently it was argued that particles with high center-of-mass energy can be produced as a result of certain particle collisions in the vicinity of a weakly magnetized non-rotating black hole [28]. Thus magnetized non-rotating black holes could serve as particle accelerators under some conditions.

Exact solutions to the Einstein-Maxwell equations provide valuable intuition for examining black hole astrophysics. Magnetized black hole solutions were constructed early in four dimensional spacetime [29]-[31] by applying Harrison transformation. Recently they were generalized to a variety of solutions to the 5D Einstein-Maxwell, and Einstein-Maxwell-dilaton equations describing black objects in external magnetic fields [32], [33]. Since only the simplest solution representing black hole on gravitational instanton, the black hole on a Kaluza-Klein bubble, has been magnetized so far [5], we consider that it is important to obtain further magnetized solutions belonging to this class.

The paper is organized as follows. In the first section we present a new exact solution to the 5D Einstein-Maxwell equations representing a static magnetized black hole on a Taub-Nut instanton. We examine its limits and obtain another solution of physical importance - a magnetized version of the Kaluza-Klein monopole. Next, we investigate the physical properties of the solution, and calculate its mass and tension using both Komar integrals and the counter-term method and comparing the results. The nut charge and potential are obtained as well, using the relations demonstrated in [14] and generalizing the definition of the nut potential appropriately for the current case. Section 4 is devoted to a rigorous derivation of the relevant Smarr relations.

2   Exact Solution

We consider the Einstein-Maxwell-dilaton gravity (EMd) in 5-dimensional spacetime with the action

$$I = \frac{1}{16\pi} \int d^5 x \sqrt{-g} \left(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - e^{-2a\varphi}F^{\mu\nu}F_{\mu\nu} \right), \tag{1}$$

which leads to the field equations

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi + 2e^{-2a\varphi}\left[F_{\mu\rho}F_\nu^\rho - \frac{1}{6}g_{\mu\nu}F_{\beta\rho}F^{\beta\rho} \right], \tag{2}$$

$$\nabla_\mu\nabla^\mu\varphi = -\frac{a}{2}e^{-2a\varphi}F_{\nu\rho}F^{\nu\rho},$$

$$\nabla_\mu \left[e^{-2a\varphi}F^{\mu\nu} \right] = 0,$$

where $R_{\mu\nu}$ is the Ricci tensor for the spacetime metric $g_{\mu\nu}$, $F_{\mu\nu}$ is the Maxwell tensor, φ is the dilaton field and a is the dilaton coupling parameter.

In the present paper we are interested in EMd solutions admitting three commuting Killing vectors, one asymptotically timelike Killing vector ξ, and two spacelike Killing vectors η and k or more precisely, solutions with a group of symmetry $R \times U(1)^2$. We focus on pure magnetic solutions with $i_\xi F = 0$ and nonzero magnetic potentials $\Phi_\eta = i_\eta F$ and $\Phi_k = i_k F$. In this case and for dilaton coupling parameter $a = \sqrt{8/3}$ we have found the following exact solution to the field equations

$$
\begin{aligned}
ds^2 &= V^{\frac{1}{3}}(r)\left[-(1-\frac{r_+}{r})dt^2 + \frac{r+r_0}{r-r_+}dr^2 + r(r+r_0)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] + \\
&\quad V^{-\frac{2}{3}}(r)\frac{r}{r+r_0}\left(d\psi + r_\infty \cos\theta d\phi\right)^2, \\
e^{-a\varphi} &= V^{\frac{2}{3}}(r), \\
\Phi_k &= -\frac{\lambda}{2}\frac{r}{r+r_0}V^{-1}(r), \\
\Phi_\eta &= \Phi_k r_\infty \cos\theta.
\end{aligned}
\tag{3}
$$

where metric function $V(r)$ is given by

$$
V(r) = \frac{1}{1+\lambda^2}\left(1 + \frac{\lambda^2 r}{r+r_0}\right).
\tag{4}
$$

Here $-\infty < \lambda < \infty$, $0 \le r_+ < \infty$, $0 \le r_0 < \infty$ are parameters and r_∞ is defined by

$$
r_\infty = \sqrt{\frac{r_0(r_0+r_+)}{1+\lambda^2}}.
\tag{5}
$$

The Maxwell 2-form F is given by

$$
F = d\psi \wedge d\Phi_k + d\phi \wedge d\Phi_\eta.
\tag{6}
$$

In the coordinates of the solution the Killing vectors are given by $\xi = \partial/\partial t$, $\eta = \partial/\partial\phi$ and $k = \partial/\partial\psi$.

As the expression reveals, the electromagnetic vector potential is directed along the 1-form corresponding to the compact dimension, which is parameterized by the angular coordinate ψ.

In the limit $\lambda \to 0$ the magnetic field vanishes and the solution reduces to the vacuum black hole on a Taub-Nut instanton [8]. It is also interesting to consider another limit by setting $r_+ = 0$. In this case we obtain a completely regular metric in the form

$$
\begin{aligned}
ds^2 &= V^{\frac{1}{3}}\left[-dt^2 + \frac{r+r_0}{r}dr^2 + r(r+r_0)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] + \\
&\quad V^{-\frac{2}{3}}\frac{r}{r+r_0}\left(d\psi + r_\infty \cos\theta d\phi\right)^2,
\end{aligned}
\tag{7}
$$

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It represents a magnetized generalization of Kaluza-Klein monopole discovered by [36], and Sorkin [37].

The solution possesses a horizon located at $r = r_+$ and its spacelike cross sections at $r = const.$ are diffeomorphic to a Hopf fibration of S^3. Taking also into account the natural limits of the solution mentioned above we can interpret our solution as a magnetized black hole on a Taub-Nut instanton.

The interval structure of the solution is the following (see fig.1):

- a semi-infinite space-like interval located at $(r \geq r_+, \theta = \pi)$ with direction $l_L = (0, r_\infty, 1)$;

- a finite timelike interval located at $(r = r_+, 0 \leq \theta \leq \pi)$ with direction $l_H = \frac{1}{\kappa_H}(1, 0, 0)$ corresponding to the black hole horizon;

- a semi-infinite space-like interval at $(r \geq r_+, \theta = 0)$ with direction $l_R = (0, -r_\infty, 1)$.

The directions of the intervals are determined by their coordinates with respect to a basis of Killing vectors $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial \psi}, \frac{\partial}{\partial \phi}\}$. The length of the S^1 fibre at infinity is equal to $L = 4\pi r_\infty$, and κ_H is the surface gravity of the horizon.
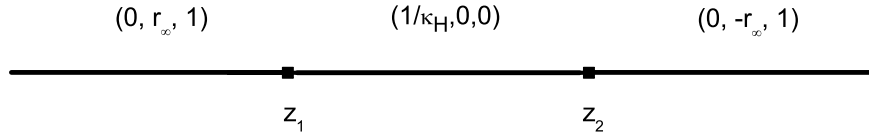

Figure 1: Interval structure of magnetized black hole on Taub-Nut instanton

Note that the parameters of the interval structure are not directly inherited from the vacuum black hole on Taub-Nut instanton since the parameter r_∞ is modified by the presence of magnetizing parameter λ.

In the spirit of the uniqueness theorem of [38], the solution is completely determined not only by its interval structure but also by an appropriately defined magnetic flux. In the case under consideration we can use the magnetic flux Ψ through the base space S^2_∞ of the S^1–fibration at infinity, namely

$$\Psi = \int_{S^2_\infty} F = L\Phi_k(\infty) = -2\pi\lambda r_\infty = -2\pi\lambda\sqrt{\frac{r_0(r_0 + r_+)}{1 + \lambda^2}}. \tag{8}$$

3   Physical Quantities

3.1   Mass and Tension

The solution is characterized by two conserved gravitational charges - the mass and the tension [39], [40], which can be calculated either by generalized Komar integrals

[41], [5], or by the counterterm method [42]. In the latter approach we consider the counterterm introduced by Mann and Stelea [43] (see also [44])

$$I_{ct} = \frac{1}{8\pi} \int d^4x \sqrt{-h} \sqrt{2\mathcal{R}}, \qquad (9)$$

leading to a boundary stress-energy tensor in the form

$$T_{ij} = \frac{1}{8\pi} \left[K_{ij} - K h_{ij} - \Omega(\mathcal{R}_{ij} - \mathcal{R} h_{ij}) - h_{ij} D^k D_k \Omega + D_i D_j \Omega \right], \qquad (10)$$

where K is the trace of the extrinsic curvature K_{ij} of the boundary, $\mathcal{R}$ and D_k are the Ricci scalar and the covariant derivative with respect to the boundary metric h_{ij}, and $\Omega = \sqrt{\frac{2}{\mathcal{R}}}$.

We will use both methods in our computations and show that they lead to equivalent results. The generalized Komar integrals for the mass and the tension are defined as

$$M_{ADM} = -\frac{L}{16\pi} \int_{S^2_\infty} \left[2 i_k \star d\xi - i_\xi \star dk \right], \qquad (11)$$

$$\mathcal{T} = -\frac{1}{16\pi} \int_{S^2_\infty} \left[i_k \star d\xi - 2 i_\xi \star dk \right],$$

where $\xi = \frac{\partial}{\partial t}$ is the Killing field associated with time translations, $k = \frac{\partial}{\partial \psi}$ is the Killing field corresponding to the compact dimension, L is the length of the S^1 fibre and S^2_∞ is the base space of S^1-fibration at infinity. By direct calculation we obtain the result

$$M_{ADM} = \frac{L}{2} \left(r_+ + \frac{1}{2} r_0 \right) \qquad (12)$$

$$\mathcal{T} = \frac{1}{4} \left(r_+ + \frac{2 + \lambda^2}{1 + \lambda^2} r_0 \right).$$

On the other hand, we can calculate the relevant components of the stress-energy tensor

$$8\pi T^t_t = \frac{1}{r^2} \left(\frac{1}{2} r_0 + r_+ \right) + \mathcal{O}(\frac{1}{r^3}), \qquad (13)$$

$$8\pi T^\psi_\psi = \frac{1}{2r^2} \left(r_+ + \frac{2 + \lambda^2}{1 + \lambda^2} r_0 \right) + \mathcal{O}(\frac{1}{r^3}).$$

According to the counterterm method the conserved quantities are obtained from the boundary stress-energy tensor as

$$\mathcal{Q} = \int_\Sigma d\Sigma_i T^i_j \xi^j, \qquad (14)$$

where ξ is a Killing vector generating an isometry of the boundary. The conserved quantity represents the mass in the case when $\xi = \partial/\partial t$, and the tension, when $\xi = \partial/\partial \psi$ [45]. Thus, we obtain

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$$M_{ADM} = \frac{1}{8\pi} \int \left(\frac{1}{2}r_0 + r_+ \right) \sin\theta d\theta d\phi d\psi, \tag{15}$$

$$\mathcal{T} = \frac{1}{16\pi} \int \left(r_+ + \frac{2+\lambda^2}{1+\lambda^2}r_0 \right) \sin\theta d\theta d\phi. $$

which leads to the same result as (12) after performing the integration. Although the expression for the ADM mass formally coincides with the corresponding one in the vacuum case [8],[10], it should be recognized that the value of the parameter L is different, since it is affected by the external magnetic field.

In addition to the ADM mass, an intrinsic mass of the black hole can be introduces by the Komar integral

$$M_H = -\frac{L}{16\pi} \int_H \left[2i_k \star d\xi - i_\xi \star dk \right]. \tag{16}$$

which in our case obtains the explicit form

$$M_H = \frac{L}{2}r_+. \tag{17}$$

The black hole mass can be expressed also in terms of the horizon area A_H and surface gravity κ_H as

$$M_H = \frac{1}{4\pi}\kappa_H A_H. \tag{18}$$

The surface gravity on the black hole horizon is determined by

$$\kappa_H = \sqrt{-\frac{1}{2}\xi_{\mu;\nu}\xi^{\mu;\nu}}|_H, \tag{19}$$

where $\xi = \partial/\partial t$ is the timelike Killing field. It leads to the result

$$\kappa_H = \frac{2\pi}{L}\sqrt{\frac{r_0}{r_+(1+\lambda^2)}} = \frac{1}{2\sqrt{r_+(r_0+r_+)}}, \tag{20}$$

The area of the horizon is calculated as

$$A_H = \int_H \sqrt{g_H}d\theta d\phi d\psi = L^2 r_+ \sqrt{\frac{r_+(1+\lambda^2)}{r_0}} = \frac{16\pi^2}{\sqrt{1+\lambda^2}}r_+^{3/2}r_0^{1/2}(r_+ + r_0). \tag{21}$$

It is obvious by the explicit expressions that (18) is satisfied.

3.2 Nut charge and potential

The spacial boundary at infinity of the solution manifold is diffeomorphic to a nontrivial S^1 bundle over S^2, therefore the solution possesses a nut charge. It is defined by the Komar-like integral [46]

$$N = -\frac{1}{8\pi} \int_{C^2} d\left(\frac{k}{\mathcal{V}}\right),\tag{22}$$

where k is the Killing 1-form associated with the S^1 fibre at infinity, $\mathcal{V}$ is its norm and C^2 is a two-dimensional surface, encompassing the nut. In our case this is equivalent to the relation

$$N = \frac{1}{2}r_\infty = \frac{L}{8\pi},\tag{23}$$

which was derived in [14] for black holes on asymptotically locally flat gravitational instantons.

In addition to the nut charge there exists a related characterstic, called a nut potential. This is revealed if we examine the 1-form $i_\xi i_k \star dk$, which can be represented in the form [14]

$$di_\xi i_k \star dk \;\; = \;\; 2\star [R(k) \wedge k \wedge \xi].\tag{24}$$

Taking into account that

$$\star R(k) = -2e^{-2a\varphi}\left(-\frac{2}{3}i_k F \wedge \star F + \frac{1}{3}F \wedge i_k \star F\right),\tag{25}$$

and using the explicit form of the electromagnetic field (6), we obtain

$$\star [R(k) \wedge k \wedge \xi] = 2d\Phi_k \wedge i_\xi i_k e^{-2a\varphi} \star F.\tag{26}$$

It follows from the field equations that $di_\xi i_k e^{-2a\varphi} \star F = 0$, consequently we can introduce an electromagnetic potential $\mathcal{B}$ such that $d\mathcal{B} = i_\xi i_k e^{-2a\varphi} \star F$. Taking advantage of it, eqn. (24) yields

$$di_\xi i_k \star dk - 4d\left(\Phi_k d\mathcal{B}\right) = 0.\tag{27}$$

The 1-form $i_\xi i_k \star dk - 4\Phi_k d\mathcal{B}$ is invariant under the Killing fields ξ, k and η and can be viewed as defined on the factor space $\hat{M} = M/R \times U(1)^2$. Since the factor space $\hat{M} = M/R \times U(1)^2$ is simply connected [47], there exists a globally defined potential χ, such as

$$d\chi = i_\xi i_k \star dk - 4\Phi_k d\mathcal{B}\tag{28}$$

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This relation determines the nut potential corresponding to the solution we investigate. It should be noted that its form distinguishes from the vacuum and electrostatic cases [14] since now it incorporates a term connected with the electromagnetic field.

The nut potential and the electromagnetic potential $\mathcal{B}$ possess the following explicit form

$$
\begin{aligned}
\chi &= \frac{r_\infty(1+\lambda^2)}{r+r_0}, \\
\mathcal{B} &= \frac{\lambda}{2}\frac{r_\infty}{r+r_0},
\end{aligned}
\tag{29}
$$

where they are normalized in such a way that they vanish at infinity.

4 Smarr-like Relations

In this section we are going to derive the relevant Smarr-like relations for the mass and the tension which provide a connection between the different characteristics of the solution. Let us consider the expression for the tension (11). It is convenient to reduce it to the factor space $\hat{M}$ by acting with the Killing field $\eta = \frac{\partial}{\partial\phi}$ associated with the azimuthal symmetry of the two-dimensional sphere at infinity [14]

$$
\mathcal{T}L = \frac{L}{8}\int_{Arc(\infty)}[i_\eta i_k \star d\xi - 2i_\eta i_\xi \star dk]
\tag{30}
$$

The integration is now performed over the semicircle representing the boundary of the two-dimensional factor space at infinity. Using the Stokes' theorem the integral can be further expanded into a bulk term over $\hat{M}$ and an integral over the rest of the boundary of the factor space, which is represented by the interval structure I_i

$$
\mathcal{T}L = \frac{L}{8}\int_{\hat{M}}[di_\eta i_k \star d\xi - 2di_\eta i_\xi \star dk] - \frac{L}{8}\sum_i \int_{I_i}[i_\eta i_k \star d\xi - 2i_\eta i_\xi \star dk].
\tag{31}
$$

If we take into account the definition of the intrinsic mass of the black hole (18) and the fact that the 1-form $i_\eta i_k \star d\xi$ vanishes along the left and right semi-infinite intervals I_L and I_R, we obtain

$$
\mathcal{T}L = \frac{1}{2}M_H + \frac{L}{4}\int_{I_L \bigcup I_R} i_\eta i_\xi \star dk + \frac{L}{8}\int_{\hat{M}}[di_\eta i_k \star d\xi - 2di_\eta i_\xi \star dk].
\tag{32}
$$

Let us consider the bulk integral and use the Ricci-identity $d\star dK = 2\star R(K)$ which applies for any Killing field K

$$
\begin{aligned}
\frac{L}{8}\int_{\hat{M}}[di_\eta i_k \star d\xi - 2di_\eta i_\xi \star dk] &= \frac{L}{8}\int_{\hat{M}}[i_\eta i_k d\star d\xi - 2i_\eta i_\xi d\star dk] = \\
&\frac{L}{4}\int_{\hat{M}}[i_\eta i_k \star R(\xi) - 2i_\eta i_\xi \star R(k)].
\end{aligned}
\tag{33}
$$

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We can further show from the field equations that for any Killing field it is satisfied

$$\star R(k) = -2e^{-2a\varphi}\left(-\frac{2}{3}i_k F \wedge \star F + \frac{1}{3}F \wedge i_k \star F\right). \tag{34}$$

Applying this relation for the Killing fields ξ and k and considering the explicit form of the electromagnetic field we obtain

$$i_\eta i_k \star R(\xi) - 2i_\eta i_\xi \star R(k) = -2\left(i_k F \wedge i_\eta i_\xi e^{-2a\varphi} \star F + i_\eta F \wedge i_k i_\xi e^{-2a\varphi} \star F\right). \tag{35}$$

Thus the bulk term becomes

$$\frac{L}{4}\int_{\hat{M}}[i_\eta i_k \star R(\xi) - 2i_\eta i_\xi \star R(k)] = -\frac{L}{2}\int_{\hat{M}}\left[i_k F \wedge i_\eta i_\xi e^{-2a\varphi} \star F + i_\eta F \wedge i_k i_\xi e^{-2a\varphi} \star F\right] =$$
$$-\frac{L}{2}\int_{\hat{M}}\left[d\Phi_k \wedge i_\eta i_\xi e^{-2a\varphi} \star F + d\Phi_\eta \wedge i_k i_\xi e^{-2a\varphi} \star F\right]. \tag{36}$$

We can further simplify the expression using Stokes' theorem and considering that the 1-form $i_k i_\xi \star F$ tends to zero at infinity, as well as that the integral over the horizon vanishes

$$-\frac{L}{2}\int_{\hat{M}}\left[d\Phi_k \wedge i_\eta i_\xi e^{-2a\varphi} \star F + d\Phi_\eta \wedge i_k i_\xi e^{-2a\varphi} \star F\right] = -\frac{L}{2}\int_{Arc(\infty)}\Phi_k i_\eta i_\xi e^{-2a\varphi} \star F -$$
$$-\frac{L}{2}\int_{I_L \bigcup I_R}\left[\Phi_k i_\eta i_\xi e^{-2a\varphi} \star F + \Phi_\eta i_k i_\xi e^{-2a\varphi} \star F\right]. \tag{37}$$

Substituting this expressions in equation (32) we obtain

$$\mathcal{T}L = \frac{1}{2}M_H + \frac{L}{4}\int_{I_L \bigcup I_R}i_\eta i_\xi \star dk - \frac{L}{2}\int_{Arc(\infty)}\Phi_k i_\eta i_\xi e^{-2a\varphi} \star F -$$
$$-\frac{L}{2}\int_{I_L \bigcup I_R}\left[\Phi_k i_\eta i_\xi e^{-2a\varphi} \star F + \Phi_\eta i_k i_\xi e^{-2a\varphi} \star F\right], \tag{38}$$

which can be also represented as

$$\mathcal{T}L = \frac{1}{2}M_H + \frac{L}{4}r_\infty \int_{I_L}d\chi - \frac{L}{4}r_\infty \int_{I_R}d\chi +$$
$$+ \frac{L}{2}\int_{I_L}(\Phi_\eta + r_\infty \Phi_k)\,d\mathcal{B} + \frac{L}{2}\int_{I_R}(\Phi_\eta - r_\infty \Phi_k)\,d\mathcal{B} -$$
$$- \frac{L}{2}\int_{Arc(\infty)}\Phi_k i_\eta i_\xi e^{-2a\phi} \star F. \tag{39}$$

From the definition (28) of the nut potential it follows that the nut potential is constant on the horizon, provided the horizon is bifurcational, and we will denote its

value by χ. Using the definition of the nut charge (23) and the fact that the nut potential vanishes at infinity the last relation is reduced to

$$
\begin{aligned}
\mathcal{T}L &= \frac{1}{2}M_H + LN\chi + \\
&+ \frac{L}{2}\int_{I_L}(\Phi_\eta + r_\infty\Phi_k)\,d\mathcal{B} + \frac{L}{2}\int_{I_R}(\Phi_\eta - r_\infty\Phi_k)\,d\mathcal{B} - \\
&- \frac{L}{2}\int_{Arc(\infty)}\Phi_k i_\eta i_\xi e^{-2a\varphi}\star F.
\end{aligned}
\tag{40}
$$

The explicit form (3) of the electromagnetic potentials Φ_k and Φ_η and the alignment of the left and right semi-infinite intervals imply that $\Phi_\eta + r_\infty\Phi_k = 0$ on I_L, and $\Phi_\eta - r_\infty\Phi_k = 0$ on I_R. Since $d\mathcal{B}$ is is regular on the factor space $\hat{M}$, the relevant integrals vanish. It remains to calculate the integral over the semicircle at infinity. We have

$$
\frac{L}{2}\int_{Arc(\infty)}\Phi_k i_\eta i_\xi e^{-2a\varphi}\star F = \frac{L}{2}\Phi_k(\infty)\int_{Arc(\infty)}i_\eta i_\xi e^{-2a\varphi}\star F = \frac{1}{2}\Psi\int_{Arc(\infty)}i_\eta i_\xi e^{-2a\varphi}\star F
\tag{41}
$$

where Ψ is the magnetic flux defined in (8).

In analogy with magnetostatics it is natural to interpret the integral

$$
J = -\frac{1}{2}\int_{Arc(\infty)}e^{-2a\varphi}i_\eta i_\xi\star F = \frac{1}{4\pi}\int_{S^2_\infty}e^{-2a\varphi}i_\xi\star F
\tag{42}
$$

as the effective current that serves as a source of the magnetic field.

Thus we obtain the Smarr-like relation for the tension in its final form

$$
\mathcal{T}L = \frac{1}{2}M_H + LN\chi + \Psi J.
\tag{43}
$$

The effective current J can be expressed via the potential Γ which is defined by $d\Gamma = e^{-2a\varphi}i_\xi i_\eta\star F$ and is given explicitly by

$$
\Gamma = -\frac{\lambda}{2}\frac{r_0\cos\theta(r - r_+)}{(1+\lambda^2)(r + r_0)},
\tag{44}
$$

where it is normalized appropriately in order to vanishes on the horizon.

It is easy to see that the effective current J is connected to the restriction of the potential Γ to the boundary of the factor space at infinity $Arc(\infty)$ as

$$
J = \frac{1}{2}\left[\Gamma(\theta = \pi)\,|_{Arc(\infty)} - \Gamma(\theta = 0)\,|_{Arc(\infty)}\right].
\tag{45}
$$

In a similar way if we take advantage of the Komar integral definition of the ADM mass we can derive the Smarr-like relation for the mass

$$
M = M_H + \frac{1}{2}LN\chi.
\tag{46}
$$

Acknowledgements

References

[1] H. Elvang and G. T. Horowitz, "When black holes meet Kaluza–Klein bubbles," Phys. Rev. D **67** (2003) 044015 [arXiv:hep-th/0210303].

[2] H. Elvang, T. Harmark and N. Obers, "Sequences of bubbles and holes: new phases of Kaluza-Klein black holes," JHEP **0501** (2005) 003 [arXiv:hep-th/0210303].

[3] S. Tomizawa, H. Iguchi and T. Mishima, "Rotating black holes on Kaluza–Klein bubbles," Phys. Rev. D **78** (2008) 084001 [arXiv:hep-th/0702207].

[4] J. Kunz and S. Yazadjiev, "Charged black holes on a Kaluza-Klein bubble," Phys. Rev. D **79** (2009) 024010 [arXiv:0811.0730 [hep-th]].

[5] S. Yazadjiev and P. Nedkova, "Magnetized configurations with black holes and Kaluza-Klein bubbles: Smarr-like relations and first law," Phys. Rev. D **80** (2009) 024005 [arXiv:0904.3605 [hep-th]].

[6] S. Yazadjiev and P. Nedkova, "Sequences of dipole black rings and Kaluza-Klein bubbles," JHEP **01** (2010) 048 [arXiv:0910.0938 [hep-th]].

[7] P. Nedkova and S. S. Yazadjiev, "Rotating black ring on Kaluza–Klein bubbles," Phys. Rev. D **82** (2010) 044010 [arXiv:1005.5051 [hep-th]].

[8] H. Ishihara and K. Matsuno, "Kaluza–Klein black holes with squashed horizons," Prog. Theor. Phys. **116** (2006) 417 [arXiv:hep-th/0510094].

[9] T. Wang, "A rotating Kaluza–Klein black hole with squashed horizons," Nucl. Phys. B **756** (2006) 86 [arXiv:hep-th/0605048].

[10] S. S. Yazadjiev, "Dilaton black holes with squashed horizons and their thermodynamics," Rhys. Rev. D **74** (2006) 024022 [arXiv:hep-th/0605271].

[11] C. Stelea, K. Schleich, D. Witt, "Non-extremal multi-Kaluza-Klein black holes in five dimesions," [arXiv:1108.5145 [gr-qc]].

[12] Y. Chen and E. Teo, "Black holes on gravitational instantons," Nucl. Phys. B **850** (2011) 253 [arXiv:1011.6464 [gr-qc]].

[13] H. Ishihara, M. Kimura, K. Matsuno, S. Tomizawa "Kaluza-Klein Multi-Black Holes in Five-Dimension Einstein-Maxwell Theory," Class. Quant. Grav. **23** (2006) 6919 [arXiv:hep-th/0605030].

[14] P. Nedkova and S. Yazadjiev, "On the thermodynamics of 5D black holes on ALF gravitational instantons," arXiv:1109.2838 [hep-th].

[15] R. Blandford, R. Znaek, "Electromagnetic extraction of energy from Kerr black holes," MNRAS **179** (1977) 433.

[16] A. N. Aliev, D. V. Galtsov, "Magnetized black holes", Phys. Usp. **32** (1989) 75.

[17] V. Karas , D. Vokrouhlicky, "On interpretation of the magnetized Kerr-Newman black hole", J. Math. Phys. **32** (1991) 714.

[18] J. Bicak, V. Karas, T. Ledvinka, "Black holes and magnetic fields", IAU Symposium No. 238 (2006), eds. V. Karas, G. Matt, Cambridge University Press, p. 139, [arXiv:astro-ph/0610841].

[19] A. Chamblin, R. Emparan and G. Gibbons, "Superconducting p-branes and extremal black holes", Phys. Rev. D **58** (1998) 084009 [arXiv:hep-th/9806017v1].

[20] R. Wald, "Black hole in a uniform magnetic field", Phys. Rev. D **10** (1974) 1680.

[21] G. Gibbons, "Black holes, magnetic fields and particle creation", MNRAS **177** (1976) 37P.

[22] A. N. Aliev, V. Frolov, "Five-dimensional rotating black hole in a uniform magnetic field: The Gyromagnetic ratio", Phys. Rev. D **69** (2004) 084022 [hep-th/0401095].

[23] A. N. Aliev, N. Ozdemir "Motion of charged particles around a rotating black hole in a magnetic field", MNRAS **336** (2002) 241, [arXiv:gr-qc/0208025].

[24] V. Frolov, A. Shoom, "Motion of charged particles near weakly magnetized Schwarzschild black hole", Phys. Rev. D **82** (2010) 084034, [arXiv:1008.2985 [gr-qc]].

[25] V. Frolov, P. Krtous, "Charged particle in higher dimensional weakly charged rotating black hole spacetime", Phys. Rev. D **83** (2011) 024016, [arXiv:1010.2266 [hep-th]].

[26] R. Konoplya, "Magnetic field creates strong superradiant instability", Phys. Lett. B **670** (2009) 459, [arXiv:0801.0846 [hep-th]].

[27] K. Kokkotas, R. Konoplya, A. Zhidenko, "Quasinormal modes, scattering and Hawking radiation of Kerr-Newman black holes in a magnetic field", Phys. Rev. D **83** (2011) 024031, [arXiv:1011.1843 [gr-qc]].

[28] V. Frolov, "Weakly magnetized black holes as particle accelerators", [arXiv:1110.6274 [gr-qc]].

[29] F. Ernst, "Black holes in a magnetic universe", J. Math. Phys. **17** (1976) 54.

[30] F. Ernst, W. Wild, "Kerr black holes in a magnetic universe", J. Math. Phys. **17** (1976) 182.

[31] A. Aliev and D. Galtsov, "Exact solutions for magnetized black holes", Astrophys. Space Sci. **155** (1989) 181.

[32] M. Ortaggio "Higher dimensional black holes in external magnetic fields," JHEP **0505** (2005) 048. [arXiv:gr-qc/0410048]

[33] S. S. Yazadjiev "Magnetized black holes and black rings in the higher dimensional dilaton gravity," Phys. Rev. D **73** (2006) 064008. [arXiv:gr-qc/0511114]

[34] G. W. Gibbons and S. W. Hawking, "Classification of gravitational instanton symmetries," Commun. Math. Phys. **66** (1979) 291.

[35] M. A. Melvin, "Pure magnetic and electric geons," Phys. Lett. **8** (1964) 65.

[36] D. J. Gross and M. J. Perry, "Magnetic monopoles in Kaluza–Klein theories," Nucl. Phys. B **226** (1983) 29.

[37] R. D. Sorkin, "Kaluza–Klein monopole," Phys. Rev. Lett. **51** (1983) 87.

[38] S. S. Yazadjiev, "A uniqueness theorem for black holes with Kaluza-Klein asymptotic in 5D Einstein-Maxwell gravity," Phys. Rev. D **82** (2010) 024015 [arXiv:1002.3954 [hep-th]].

[39] J. H. Traschen, D. Fox, "Tension perturbations of black brane space-times," Class. Quant. Grav. **21** (2004) 289 [gr-qc/0103106].

[40] J. H. Traschen, "A Positivity theorem for gravitational tension in brane space-times," Class. Quant. Grav. **21** (2004) 1343 [hep-th/0308173].

[41] P. Townsend and M. Zamaklar, "The first law of black brane mechanics," Class. Quant. Grav. **18** (2006) 5269 [arXiv:hep-th/0511180].

[42] V. Balasubramanian and P. Kraus, "A Stress Tensor for Anti-de Sitter Gravity," Commun. Math. Phys. **208** (1999) 413 [arXiv:hep-th/9902121].

[43] R. Mann and C. Stelea, "On the gravitational energy of the Kaluza Klein monopole," Rhys. Lett. B **634** (2006) 531 [arXiv:hep-th/0511180].

[44] D. Astefanesei and E. Radu, "Quasilocal formalism and black ring thermodynamics," Rhys. Rev. D **73** (2006) 044014 [arXiv:hep-th/0509144].

[45] C. Stelea, K. Schleich and D. Witt, "Charged Kaluza-Klein double-black holes in five dimensions," Rhys. Rev. D **83** (2011) 084037 [arXiv:0909.3835].

[46] C. J. Hunter, "The action of instantons with nut charge," Phys. Rev. D **59** (1999) 024009 [arXiv:gr-qc/9807010].

[47] S. Hollands and S. Yazadjiev, "Uniqueness theorem for 5-dimensional black holes with two axial Killing fields," Commum. Math. Phys. **283** (2008) 749 [arXiv:0707.2775 [gr-qc]].